A Robust Approach for the Derivation of Closed-Form Green's Functions

M. I. Aksun, *Member, IEEE*

Abstract— **Spatial-domain Green's functions for multilayer, planar geometries are cast into closed forms with two-level approximation of the spectral-domain representation of the Green's functions. This approach is very robust and much faster compared to the original one-level approximation. Moreover, it does not require the investigation of the spectral-domain behavior of the Green's functions in advance to decide on the parameters of the approximation technique, and it can be applied to any component of the dyadic Green's function with the same ease.**

I. Introduction

NUMERICAL modeling of printed structures used in monolithic millimeter and microwave integrated circuits (MMIC) can be efficiently and rigorously performed by employing the method of moments (MoM). The MoM is based upon the transformation of an operator equation, such as integral, differential, or integro-differential operators, into a matrix equation [1]. Although the MoM is the most efficient numerical technique for moderate-size printed geometries (spanning several wavelengths in two dimensions), there is still need for improvement, which could be accomplished in the calculation of the matrix elements and in the solution of the matrix equation. For small geometries like those requiring couple hundreds of unknowns, the matrix-fill time could be the significant part of the overall solution time, however, for large geometries the matrix solution time will dominate the CPU time [2].

In the application of the spatial-domain MoM to the solution of a mixed-potential integral equation (MPIE), one needs to calculate the Green's functions of the vector and scalar potentials in the spatial domain where they are represented as oscillatory integrals, called Sommerfeld integrals. The evaluation of these integrals is quite time consuming, therefore the matrix-fill time would be significantly improved if these integrals can be evaluated efficiently. Recently, a technique has been proposed to approximate these integrals analytically for a horizontal electric dipole over a thick substrate backed by a ground plane; this is called the closed-form Green's functions method [3]. This technique was improved first for two layer geometries with arbitrary thicknesses [4], then for multilayer geometries with horizontal electric dipole (HED), horizontal magnetic dipole (HMD), vertical electric dipole (VED), and vertical magnetic dipole (VMD) sources [5]. However, a question remains to be answered on the robustness and the efficiency of the technique, because some of the Green's functions are usually difficult to approximate and it is recommended that the function to be approximated needs to be examined in advance. The source of difficulties in this technique is the approximation of the spectral-domain Green's functions in terms of complex exponentials. The originally proposed technique [3] uses the original Prony method which requires the same number of samples as the number of unknowns, that is, the number of samples must be twice as many as the number of complex exponentials (one for the coefficient and one for the exponent). Therefore, it would be difficult to account for the fast changes in the spectral domain without using tens of complex exponentials if not hundreds in certain cases, which is partly due to the uniform sampling required by the Prony method. The use of the least-square Prony method has improved the technique to account for the fast changes with a reasonable number of exponentials [4], but due to the noise sensitivity of the Prony methods [6], [7], it requires several trial and error iterations which render the technique to be inefficient and not robust. As a solution, another approximation technique, called the generalized pencil of function (GPOF) method [8], is employed in casting the Green's functions into closed forms [5]. The GPOF method has turned out to be quite robust and less noise sensitive when compared to the original and least-square Prony's methods, and also provides a good measure for choosing the number of exponentials used in the approximation. However, it still requires one to study in advance the spectral-domain behavior of the Green's function in order to decide on the approximation parameters like the number of sampling points and the maximum value of the sampling range. In addition, since the approximation techniques, like the Prony and the GPOF methods, require the function to be sampled uniformly, one would need to take hundreds of samples in order to be able to approximate a slow converging function with rapid changes (even if this were to occur in a small region), which is a typical behavior of the spectral-domain Green's functions of the scalar potentials in a thin substrate. Because of these difficulties, the technique of deriving the closed-form Green's functions and subsequently using them in MoM applications are considered to be not robust and could not be used much for the development of a general-purpose electromagnetic

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The author is with the Department of Electrical and Electronics Engineering, Bilkent University, 06533 Ankara, Turkey.

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software. In this paper, a new approach based on a two-level approximation is proposed to overcome these difficulties, and demonstrated that it is very robust and computationally efficient.

The procedure of the original one-level approximation is described and difficulties associated with this approach are demonstrated on some examples by using the GPOF method in Section II of this paper. This is followed in Section III where the formulation of the new approach based on a two-level approximation and some numerical examples are included. Then, in Section IV, a discussion on the new technique and conclusions are provided.

II. Difficulties in the Original One-Level Approximation

Since the main goal of this paper is to introduce a robust technique to obtain the spatial-domain Green's functions in closed forms for planarly-layered media, Fig. 1, it would be useful to give the definition of the spatial-domain Green's functions

$$G = \frac{1}{4\pi}\int_{SIP} dk_\rho k_\rho H_o^{(2)}(k_\rho\rho)\tilde{G}(k_\rho) \tag{1}$$

where, G and $\tilde{G}$ are the Green's functions in the spatial and spectral domains, respectively, $H_o^{(2)}$ is the Hankel function of the second kind and SIP is the Sommerfeld integration path defined in Fig. 2. Note that this integral, called the Sommerfeld integral, can not be evaluated analytically for the spectral-domain Green's functions $\tilde{G}$, which are obtained analytically for planarly stratified media [5], [9]. It was recognized by Chow *et al.* [3] that if the spectral-domain Green's function $\tilde{G}$ is approximated by exponentials, the Sommerfeld integral (1) can be evaluated analytically using the well-known Sommerfeld identity

$$\frac{e^{-jkr}}{r} = -\frac{j}{2}\int_{SIP} dk_\rho k_\rho H_o^{(2)}(k_\rho\rho)\frac{e^{-jk_z|z|}}{k_z}. \tag{2}$$

Therefore, this places the emphasis of deriving the closed-form Green's functions on the exponential approximation. Since the approximation techniques used for this problem, namely the original Prony, the least square Prony and the GPOF methods, require uniform samples along a real variable of a complex-valued function, one might think of choosing the integration path in (1) along the real k_ρ axis so that $\tilde{G}$ can be sampled along a real variable. However, one should notice that $k_z^2 = k^2 - k_\rho^2$ and sampling along real k_ρ results in an approximation in terms of exponentials of k_ρ which cannot be cast into a form of exponentials of k_z as required in the application of the Sommerfeld identity (2). Hence, a deformed path on k_ρ plane, denoted by C_{ap} in Fig. 2, was defined as a mapping of a real variable t onto the complex k_z plane by

$$k_z = k\left[-jt + \left(1 - \frac{t}{T_o}\right)\right], \qquad 0 \le t \le T_o \tag{3}$$

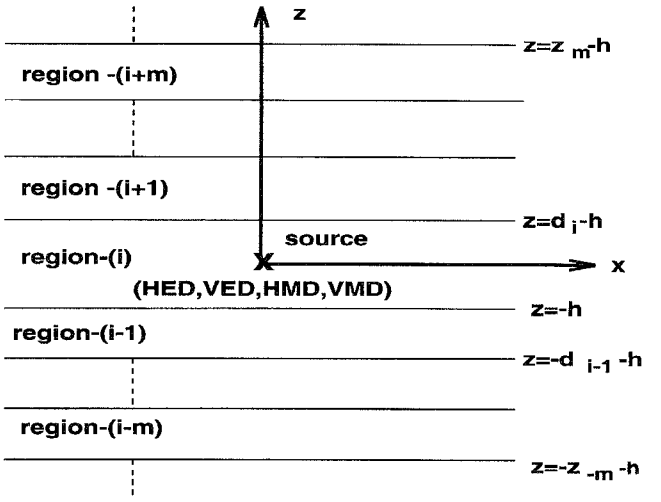


Fig. 1. A typical planar geometry.

where k_z and k are defined in the source layer [3]. The Green's functions are sampled uniformly on $t \in [0, T_o]$, which maps onto the path C_{ap} with $k_{\rho_{\max}} = k[1 + T_o^2]^{1/2}$ in the k_ρ-plane, and approximated in terms of exponentials of t which can easily be transformed into a form of exponentials of k_z. This scheme is called the one-level approximation approach here in this paper because the complex function to be approximated is sampled between zero and T_o and is assumed to be negligible beyond T_o.

For a general-purpose algorithm, the spectral-domain Green's functions are obtained for a multilayer medium and neither surface-wave poles nor the real images are extracted. It is true that the extraction of the surface-wave poles (SWP) and the real images would have helped the exponential approximation techniques by making the Green's functions in the spectral domain well-behaving (extraction of the SWP's) and fast-converging (extraction of the real images). However, since the contribution of the SWP's is small for geometries on a thin substrate, and there is no analytical way of finding the real images for multilayer planar structures except for simple cases like single and double layers, the help gained for the approximation would be limited to a restricted class of planar geometries and would render the algorithm not general purpose and not robust.

It would be instructive to consider the practical details of the implementation of the exponential approximation along the path defined in (3). It is of utmost importance to choose the approximation parameters; T_o, the number of exponentials to be used in the approximation, and the number of samples in $t \in [0, T_o]$, judiciously for the success of this approach. To illustrate the implementation of the one-level exponential approximation and the difficulties involved, the spectral-domain Green's function for the scalar potential due to an x-directed dipole, $\tilde{G}_x^q$, is given in Fig. 3, for a geometry of four layers at 30 GHz: First layer-PEC; second layer-$\epsilon_{r2} = 12.5, d_2 = 0.03$ cm; third layer-$\epsilon_{r3} = 2.1, d_3 = 0.07$ cm; fourth layer-free-space, and the source and observation planes are chosen at the interface of the second and third layers. Since the expressions of the spectral-domain Green's functions in a

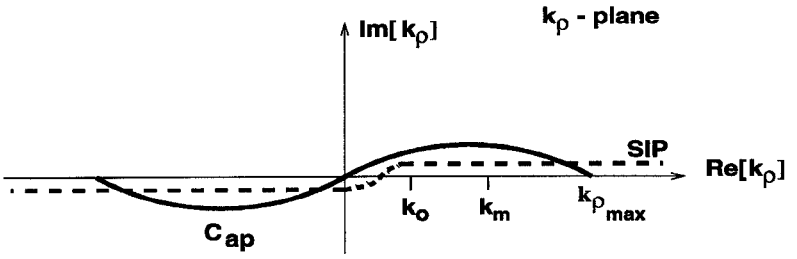


Fig. 2. Definition of the Sommerfeld integration path and the path C_{ap} used in one-level approximation.

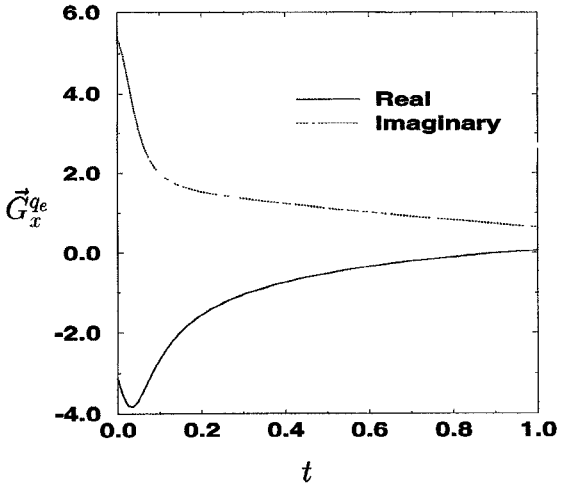


Fig. 3. The magnitude of the spectral-domain Green's function $\tilde{G}_x^q$ along the path C_{ap}. First layer-PEC; second layer-$\epsilon_{r2} = 12.5, d_2 = 0.03$ cm; third layer-$\epsilon_{r3} = 2.1, d_3 = 0.07$ cm; fourth layer-free-space, freq = 30 GHz.

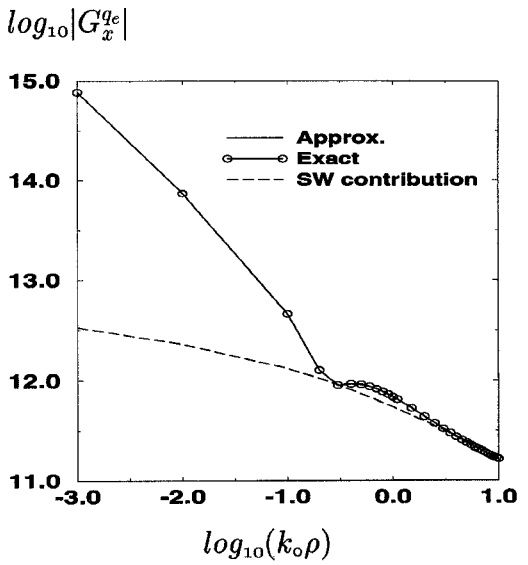


Fig. 4. The magnitude of the Green's function for the scalar potential and the surface wave contribution. First layer-PEC; second layer-$\epsilon_{r2} = 12.5, d_2 = 0.03$ cm; third layer-$\epsilon_{r3} = 2.1, d_3 = 0.07$ cm; fourth layer-free-space, freq = 30 GHz.

multilayer medium are given in [5] for HED, VED, HMD, and VMD sources, they are not included in this paper. It is evident from Fig. 3 that Green's functions can have sharp peaks and fast changes for small t, which maps to the far-field region in the spatial domain. Therefore, one needs to sample the Green's function given in Fig. 3 at a period of less than 0.05 along t so that the fine features of the function can be captured in the approximation. The choice of T_o is another parameter that competes with the period of samples because large T_o corresponds to large number of samples and translates to a longer CPU time. Fortunately, for the example given in Fig. 3, the Green's function decays quite fast in the spectral domain, therefore it would be enough to sample as far as $T_o = 5$ which requires 200 samples if Δt is chosen to be 0.025. The spatial-domain Green's function is obtained via the GPOF method using the above approximation parameters ($T_o = 5$, number of samples = 201, number of exponentials = 13) and compared to the result obtained from the numerical integration, which are labeled as "Apprx." and "Exact," respectively, in Fig. 4. Although, as it was mentioned above, the SWP's are not extracted from the spectral-domain Green's function prior to the exponential approximation, the contribution of the SWP's is also shown for the purpose of comparison and one can draw a conclusion that the exponential approximation algorithm (GPOF) works fine well within the influence range of the SWP's and beyond that an asymptotic approximation together with the surface-wave contribution can be used to approximate the spatial-domain Green's functions [10], [11].

Unfortunately, not all the Green's functions have fast decaying spectral-domain behavior like the above example given in Fig. 3. For example, the spectral-domain Green's function for the vertical component of the vector potential due to a HED [5], $\tilde{G}_{zx}^A/jk_x = \tilde{G}_{zy}^A/jk_y$, does not decay as fast and moreover has a relatively sharp peak which requires sampling almost as frequently as that of the example given in Fig. 3, as shown in Fig. 5. To demonstrate the effect of the approximation parameters, the Green's function $\int G_{zx}^A\, dx\, (= \mathcal{F}^{-1}\{\tilde{G}_{zx}^A/jk_x\})$ is given for the same approximation parameters as those of the above example ($T_o = 5$, number of samples = 201) and compared to the results obtained by the numerical integration of the spectral-domain representation of the Green's function and to the results obtained by using different approximation parameters in Fig. 6. It is observed that the approximated Green's functions do not agree with the exact solution for small values of ρ because the spectral-domain Green's function is not sampled far enough to get an accurate near-field distribution. However, if the value of T_o is increased, the agreement between the approximated and

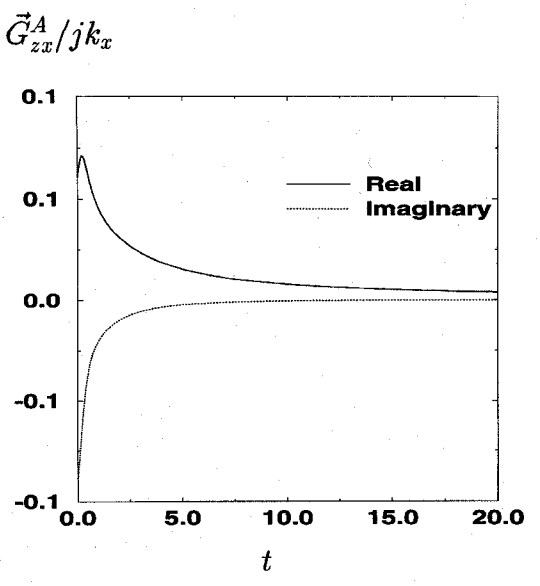


Fig. 5. The magnitude of the spectral-domain Green's function $\tilde{G}^A_{zx}/jk_x$. First layer-PEC; second layer-$\epsilon_{r2} = 12.5, d_2 = 0.03$ cm; third layer-$\epsilon_{r3} = 2.1, d_3 = 0.07$ cm; fourth layer-free-space, freq = 30 GHz.

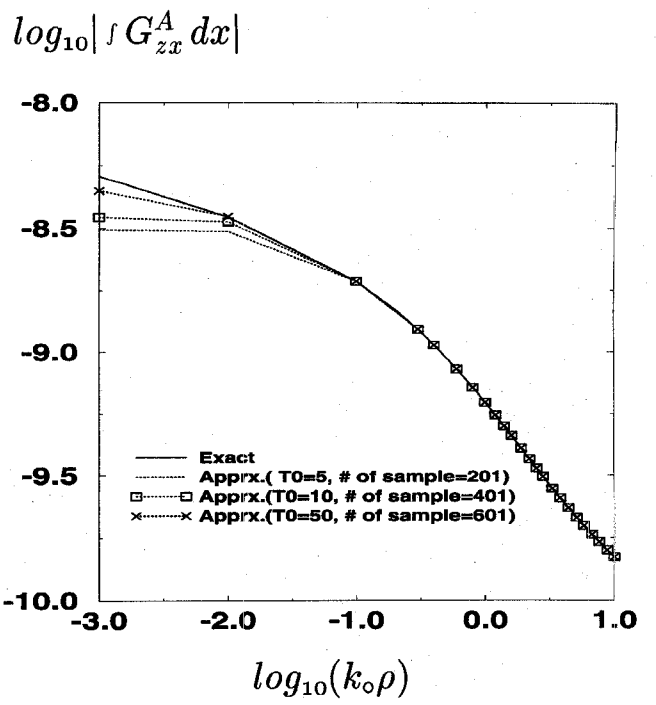


Fig. 6. The magnitude of the Green's function for the vector potential $\int G^A_{zx}\,dx$. First layer-PEC; second layer-$\epsilon_{r2} = 12.5, d_2 = 0.03$ cm; third layer-$\epsilon_{r3} = 2.1, d_3 = 0.07$ cm; fourth layer-free-space, freq = 30 GHz.

exact Green's functions is improved at the expense of the computation time provided that the frequency of sampling is kept constant.

From the above discussion, it can easily be concluded that the one-level approximation approach can not be made fully robust and suitable for the development of CAD software. As it was mentioned above, this is because it requires users first to investigate the spectral-domain behavior of the Green's function and then to perform a few iterations to find the best possible combination of the approximation parameters. To circumvent these difficulties, a two-level approximation scheme is developed here in conjunction with the use of the GPOF method and its details are given in the following section.

III. Two-Level Approach for Approximating the Spectral-Domain Green's Functions

To alleviate the necessity of investigating the spectral-domain Green's functions in advance and the difficulties caused by the trade-off between the sampling range T_o and the sampling period, the approximation is performed in two levels. The first part of the approximation is performed along the path C_{ap1} while the second part is done along the path C_{ap2}, as shown in Fig. 7. Note that the second part of the approximation is the same as the one-level approximation scheme described in the previous section, except that now the value of T_{o2} ($k_{\rho_{max2}} = k[1 + T_{o2}^2]^{1/2}$) can be set in advance to a value such that $k_{\rho_{max2}} \geq k_m$ where k_m is the maximum value of the wavenumber involved in the geometry.

To illustrate the procedure of the two-level approximation, we will first outline the necessary steps and then provide some of the details. The steps are:

1) Choose T_{o2} such that $k_{\rho_{max2}} \geq k_m$: For example, since GaAs is the highest dielectric constant layer (ϵ_r(GaAs) = 12.5), then $k_m = \sqrt{12.5}k_o$, and T_{o2} can be safely chosen to be five.
2) Choose T_{o1}, i.e., $k_{\rho_{max1}} = k[1 + (T_{o1} + T_{o2})^2]^{1/2}$, and the number of samples on $[k_{\rho_{max2}}, k_{\rho_{max1}}]$: The choice of T_{o1} is not very critical as long as one chooses $k_{\rho_{max1}}$ large enough to pick up the behavior of the spectral-domain Green's function for large k_ρ, and, since the spectral-domain behaviors of the Green's functions are always smooth beyond $k_{\rho_{max2}}$, it is not necessary to have a large number of samples on $[k_{\rho_{max2}}, k_{\rho_{max1}}]$. Typical values could be 200 for T_{o1} and 200 for the number of samples.
3) Sample the function along the path C_{ap1} and approximate it by using the GPOF method: Sampling along the path C_{ap1} can be performed by varying t between zero and T_{o1} uniformly in $k_z = -jk[T_{o2} + t]$.
4) Subtract the function approximated for the range of $k_\rho \in [k_{\rho_{max2}}, k_{\rho_{max1}}]$ from the original function: The remaining function will be nonzero over a small range of k_ρ ($\in [0, k_{\rho_{max2}}]$) so that one can pick up the fine features of this function without employing a huge number of sampling points.
5) Sample the remaining function uniformly along the path C_{ap2} and approximate it by using the GPOF method: Sampling along the path C_{ap2} can be performed by varying t between zero and T_{o2} uniformly in $k_z = k[-jt + (1 - t/T_{o2})]$.

The parameters that must be fixed by the user in advance are the limits of the sampling ranges T_{o1} and T_{o2} for the first and the second parts of the approximation, respectively, and the number of samples along the paths C_{ap1} and C_{ap2}, which respectively correspond to the first and second parts of the approximation. Although the number of parameters which are

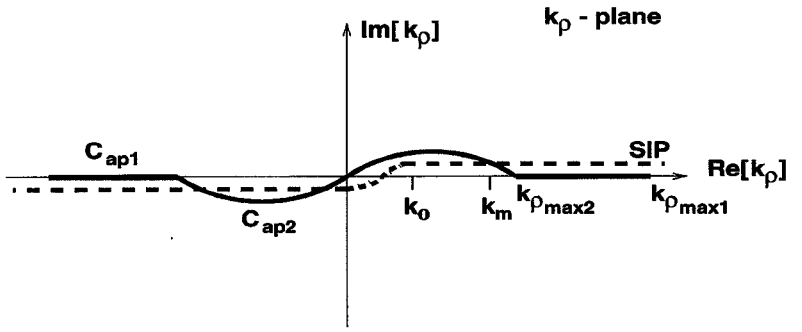


Fig. 7. The paths C_{ap1} and C_{ap2} used in two-level approximation.

to be decided by the user seems to have increased compared to the one-level approximation, these parameters are determined only once for the class of geometries that are of interest; they are used for the approximation of any component of the dyadic Green's function and for any geometrical constants. Moreover, the choice of these parameters do not require an investigation of the function to be approximated in advance because they can be chosen for the possible limits of the geometrical constants.

To demonstrate the robustness of the technique, the choice of the parameters and the application of the above procedure, the Green's function $\int G_{zx}^A\, dx$ is obtained for the same geometry given in Section II. Its spectral-domain representation is given here as

$$\frac{\tilde{G}_{zx}^A}{jk_x} = \frac{-\mu_i}{2jk_{z_i}}\left[\frac{k_{z_i}}{k_\rho^2}(A(k_{z_0},\cdots,k_{z_N})+B(k_{z_0},\cdots,k_{z_N}))e^{jk_{z_i}z} + \frac{k_{z_i}}{k_\rho^2}(C(k_{z_0},\cdots,k_{z_N})+D(k_{z_0},\cdots,k_{z_N}))e^{-jk_{z_i}z}\right] \tag{4}$$

to help explain the approximation procedure, where the layer "i" denotes the source layer, and A, B, C, and D are given in [5]. It should also be noted that this expression is for the case where the source and observation points are in the same layer, i.e., layer "i." If it is desired to find the Green's function for the observation layer different from the source layer, then the coefficients of the up-going waves and down-going waves must be carried to the observation layer with a recursive algorithm [5], [9]. Let us first give the parametric equations describing the paths C_{ap1} and C_{ap2} for the first and second parts of the approximation, respectively

$$\text{For } C_{ap1} \quad k_{z_i} = -jk_i[T_{o2}+t] \qquad 0 \le t \le T_{o1} \tag{5}$$

$$\text{For } C_{ap2} \quad k_{z_i} = k_i\left[-jt+\left(1-\frac{t}{T_{o2}}\right)\right] \qquad 0 \le t \le T_{o2} \tag{6}$$

where t is the running variable sampled uniformly on the corresponding range. Then, the above procedure is followed step-by-step as:

1) $T_{o2} = 5$ is chosen, for which $k_{\rho_{\max 2}} = k_i[1 + T_{o2}^2]^{1/2} > k_m = \sqrt{12.5}\, k_o$.
2) $T_{o1} = 400$ is chosen to ensure that the behavior of $\tilde{G}_{zx}^A/jk_x$ for large k_ρ is captured. This choice is not critical, 300 or 500 could have been chosen instead. Since there is no fine feature to pick up in this range, that is, the function is smooth, one can keep the range large without having to use a large number of samples. Therefore, the number of samples is chosen to be 50.
3) $\tilde{G}_{zx}^A/jk_x$ is sampled along the path C_{ap1} and the GPOF method is applied

$$f(k_\rho)\left(\cong \frac{\tilde{G}_{zx}^A}{jk_x} \text{ for } k_\rho \in [k_{\rho_{\max 2}}, k_{\rho_{\max 1}}]\right) = \sum_{n=1}^{N_1} b_{1n}e^{\beta_{1n}t} = \sum_{n=1}^{N_1} a_{1n}e^{-\alpha_{1n}k_{z_i}} \tag{7}$$

$$\alpha_{1n} = \frac{\beta_{1n}}{jk_i}; \qquad a_{1n} = b_{1n}e^{-jk_i\alpha_{1n}T_{o2}} \tag{8}$$

where b_{1n} and β_{1n} are coefficients and exponents obtained from the GPOF method, and N_1 is the number of exponentials used in this approximation. The choice of the number of exponentials is based upon the number of significant singular values obtained in an intermediate step of the application of the GPOF method. For this specific problem, five exponentials are chosen to approximate the Green's function on the range of $k_\rho \in [k_{\rho_{\max 2}}, k_{\rho_{\max 1}}]$. The transformation of the coefficients b_{1n} and the exponents β_{1n} is necessary to cast the approximating function into a form suitable for the application of the Sommerfeld identity (2), that is, the approximating function must be an exponential function of k_{z_i}. Hence, a_{1n} and α_{1n} are obtained in terms of b_{1n} and β_{1n} in (8)
4) The approximating function $f(k_\rho)$ is subtracted from the original function $\tilde{G}_{zx}^A/jk_x$, which guarantees the remaining function to be negligible beyond $k_{\rho\max 2}$

$$G = \frac{1}{4\pi}\int_{C_{ap2}} dk_\rho k_\rho H_o^{(2)}(k_\rho\rho)\left[\frac{\tilde{G}_{zx}^A(k_\rho)}{jk_x} - f(k_\rho)\right] + \frac{1}{4\pi}\int_{C_{ap2}+C_{ap1}} dk_\rho k_\rho H_o^{(2)}(k_\rho\rho)f(k_\rho). \tag{9}$$

Note that the first integral is evaluated along the path C_{ap2} because the integrand is negligible on C_{ap1}, but

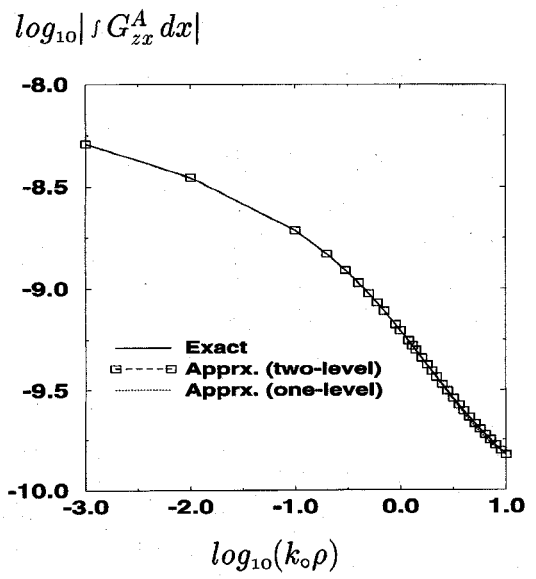


Fig. 8. The magnitude of the Green's function for the vector potential $\int G^A_{zx}\,dx$. First layer-PEC; second layer-$\epsilon_{r2} = 12.5, d_2 = 0.03$ cm; third layer-$\epsilon_{r3} = 2.1, d_3 = 0.07$ cm; fourth layer-free-space, freq = 30 GHz.

the second integral is evaluated along $C_{ap2} + C_{ap1}$. Therefore, the Sommerfeld identity (2) can be applied to the integrals in (9).

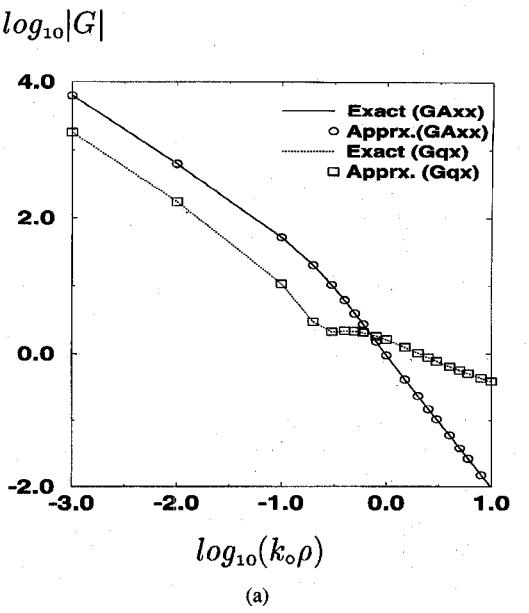
5) The remaining function is sampled along the path C_{ap2} with 100 samples. Since the maximum range for the sampling ($k_{\rho_{max2}}$) is rather small, compared to that of the one-level approximation scheme, the frequency of sampling can be made quite high without substantially increasing the number of samples. For all practical purposes (including the worst case situation) the choice of 200 as the number of samples would be more than enough to get a good approximation

$$\frac{\tilde{G}^A_{zx}}{jk_x} - f(k_\rho) \cong \sum_{n=1}^{N_2} b_{2n} e^{\beta_{2n} t} = \sum_{n=1}^{N_2} a_{2n} e^{-\alpha_{2n} k_{z_i}} \quad (10)$$

$$\alpha_{2n} = \frac{\beta_{2n} T_{o2}}{k_i(1 + jT_{o2})}; \quad a_{2n} = b_{2n} e^{k_i \alpha_{2n}} \quad (11)$$

where b_{2n} and β_{2n} are the coefficients and exponents of the exponentials of t obtained from the application of the GPOF method, and a_{2n} and α_{2n} are the coefficients and exponents of the exponentials of k_{z_i}. The number of exponentials N_2 in this part of the approximation is chosen to be 8, again by the number of significant singular values.

To summarize, the approximation parameters as chosen here are as follows: for the first part of the approximation; $T_{o1} = 400, T_{o2} = 5$, number of samples = 50, and number of exponentials = 5, for the second part of the approximation; number of samples = 100, number of exponentials = 8. Note that the total number of exponentials used in this approximation is 13. The Green's function obtained by employing the above procedure is given in Fig. 8 along with the data obtained from direct numerical evaluation of the Sommerfeld-type integral (exact), and from the one-level approximation approach with the parameters of approximation $T_o = 200$, number of samples = 400 and the number of exponentials = 13. Note that the values of the parameters used in the one-level approximation are chosen to make the computation time minimum with a reasonable agreement. However, those of the two-level approximation are typical values and the number of samples for the second part of the approximation can even be reduced to 50 with no change in the results. The two approximation techniques for the above example are compared for the CPU time on a SPARCstation 10/41, using



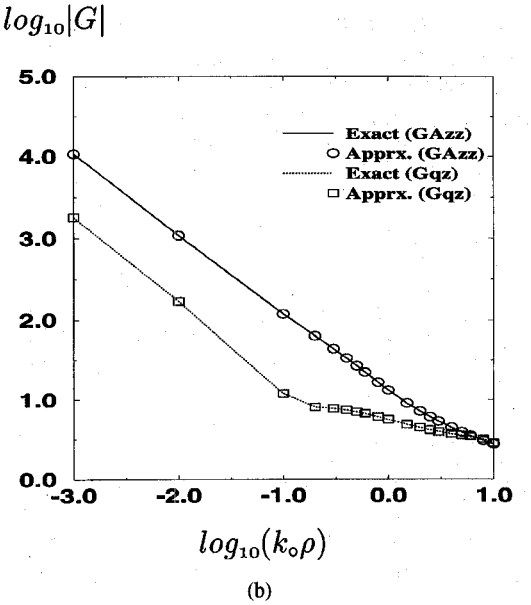
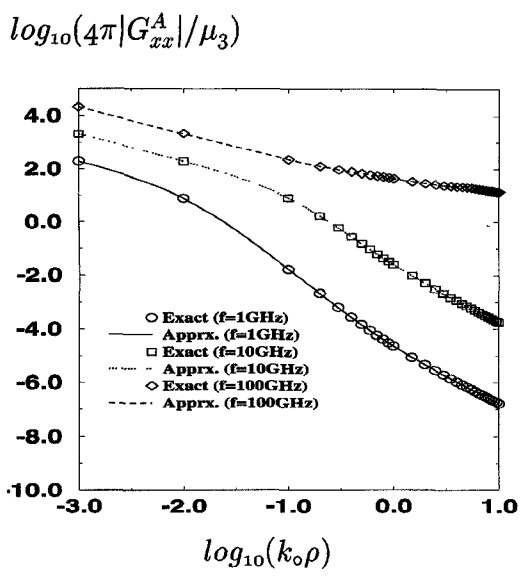
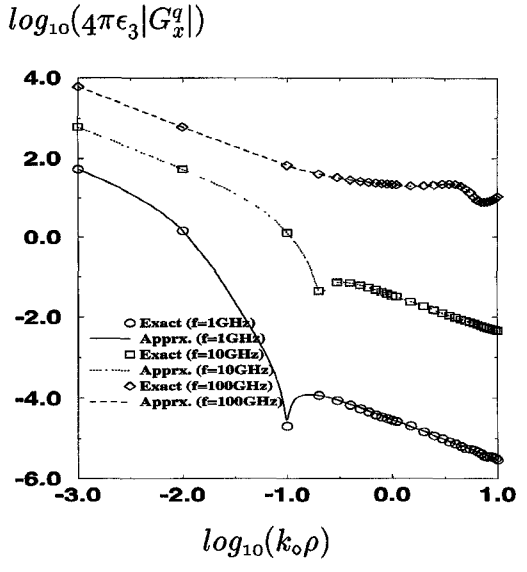


Fig. 9. (a) The magnitude of the normalized Green's functions $4\pi G^A_{xx}/\mu_3, 4\pi\epsilon_3 G^q_x$. First layer-PEC; second layer-$\epsilon_{r2} = 12.5, d_2 = 0.03$ cm; third layer-$\epsilon_{r3} = 2.1, d_3 = 0.07$ cm; fourth layer-free-space, freq = 30 GHz. (b) The magnitude of the normalized Green's functions $4\pi G^A_{zz}/\mu_3, 4\pi\epsilon_3 G^q_z$. First layer-PEC; second layer-$\epsilon_{r2} = 12.5, d_2 = 0.03$ cm; third layer-$\epsilon_{r3} = 2.1, d_3 = 0.07$ cm; fourth layer-free-space, freq = 30 GHz.



(a)



(b)

Fig. 10. (a) The magnitude of the normalized Green's function $4\pi G^A_{xx}/\mu_3$. First layer-PEC; second layer-$\epsilon_{r2} = 12.5, d_2 = 0.03$ cm; third layer-$\epsilon_{r3} = 2.1, d_3 = 0.07$ cm; fourth layer-free-space. (b) The magnitude of the normalized Green's function $4\pi\epsilon_3 G^q_x$. First layer-PEC; second layer-$\epsilon_{r2} = 12.5, d_2 = 0.03$ cm; third layer-$\epsilon_{r3} = 2.1, d_3 = 0.07$ cm; fourth layer-free-space.

the same number of total exponentials (= 13) for different approximation parameters, and presented in the table format below:

Approximation	Approximation Parameters	CPU time (sec)
one-level	$T_o = 200, N_s = 400$	198.0
one-level	$T_o = 200, N_s = 500$	382.0
two-level	$T_{o1} = 400, N_{s_1} = 50$ $T_{o2} = 5, N_{s_2} = 50$	1.2
two-level	$T_{o1} = 400, N_{s_1} = 50$ $T_{o2} = 5, N_{s_2} = 100$	3.5

where N_s is the number of samples in one-level approximation scheme while N_{s1} and N_{s2} are the number of samples of the first and second parts of the approximation, respectively, in the two-level approximation approach. It is obvious that the two-level approximation approach improves the computational efficiency significantly.

The robustness of the two-level approach can be demonstrated by casting the other Green's functions into closed forms with the use of the same approximation parameters as those used for $\int G^A_{zx}\, dx$, namely $T_{o1} = 400, N_{s_1} = 50$ $T_{o2} = 5, N_{s_2} = 100$. The normalized Green's functions of the vector and scalar potentials due to HED and VED sources are obtained ($4\pi G^A_{xx}/\mu_3, 4\pi\epsilon_3 G^q_x, 4\pi G^A_{zz}/\mu_3$ and $4\pi\epsilon_3 G^q_z$) following the two-level approach (Apprx.) and evaluating the Sommerfeld integrals numerically (Exact), and given in Fig. 9(a) and (b). This test shows that the same set of approximation parameters can be used for any Green's function, that is, there is no need for an advance investigation of the Green's function and no need for any trial steps. The assessment of the robustness of the proposed approach also requires a study of the sensitivity of the approximation parameters to the geometrical constants and the frequency. Therefore, the Green's functions for the vector and the scalar potentials are obtained in closed forms for the same geometrical constants and for the same approximation parameters used above, but the frequency of operation is changed to 1 GHz, 10 GHz and 100 GHz, which is equivalent, in effect, to a change of the geometrical constants, Fig. 10(a) and (b). It is observed that the agreements between the exact and approximate sets of data are still perfect and hence it is safe to conclude that the two-level approach proposed in this paper is very robust.

IV. Conclusion

The closed-form Green's functions developed previously suffer from the difficulties of choosing approximation parameters for the exponential approximation techniques used in the derivation, thereby rendering the technique to be inefficient and not robust. Moreover, the extraction of the SWP's and real images may not be possible or efficient for multilayer geometries when the original approach is used. Here, a new approach based on a two-level approximation is proposed to overcome these difficulties and to make the use of closed-form Green's functions attractive for those developing EM software and for researchers in the field. The major advantages of this approach are its robustness and the computational efficiency, both of which are demonstrated in the text.

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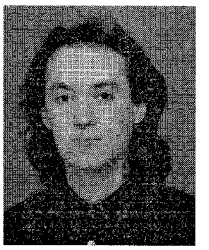

M. I. Aksun (M'92) received the B.S. and M.S. degrees in electrical and electronics engineering from the Middle East Technical University, Ankara, Turkey, in 1981 and 1983, respectively, and the Ph.D. degree in electrical and computer engineering from the University of Illinois at Urbana-Champaign in 1990.

From 1990 to 1992, he was a post doctoral fellow at the Electromagnetic Communication Laboratory, University of Illinois at Urbana-Champaign. Since 1992, he has been on the faculty of the Department of Electrical and Electronics Engineering at Bilkent University, Ankara, Turkey, where he is currently an Associate Professor. His research interests include the numerical methods for electromagnetics, microstrip antennas, and microwave and millimeter-wave integrated circuits.